ISSN 2348-1218 (print)
International Journal of Interdisciplinary Research and Innovations ISSN 2348-1226 (online)
Vol. 2, Issue 3, pp: (9-13), Month: July 2014 - September 2014, Available at: www.researchpublish.com

DOUBLE INTEGRAL REPRESENTATION AND CERTAIN TRANSFORMATIONS FOR BASIC APPELL FUNCTIONS

Farooq Ahmad[1], D.K.Jain[2], Renu Jain[3]

[1,3]School of Mathematics and Allied Sciences, Jiwaji University, Gwalior 474 011 (M.P.), India
[2]Department of Applied Mathematics, Madhav Institute of Technology and Science, Gwalior 474 005 (M.P.), India

Abstract: **In the present paper we have studied basic analogue of Appell's hypergeometric functions called q-Appell functions and expressed these functions ϕ^1, ϕ^2, ϕ^3 in terms of definite integrals. Also certain transformation formulae have been obtained related to these functions. Some special cases have been also discussed.**
Mathematics Subject Classification: 33D60, 41A60, 33C65.

Keywords: **q-Appell functions, Integral representation, transformation formulae.**

1. INTRODUCTION

The basic analogue of Appell's hypergeometric functions of two variables were defined and studied by Jackson [1]. Agarwal [2] also studied these functions and gave some general identities involving these functions. Andrews [3] also worked upon these functions and showed that the first of the Appell series can be reduced to a $_3\Phi_2$ series.

Bhaskar and Shrivastava also defined bibasic Appell series and obtained summation formulae, integral representation and continued fractions for these functions .Yadav and Purohit [4] employed the q-fractional calculus approach to derive a number of summation formulae for the generalized basic hypergeometric functions of one and more variables in terms of the q-gamma functions.

Apart from Jackson's initial work, Agarwal developed some properties of basic Appell series and Slater [5] applied contour integral techniques to such series and observed that there was apparently no systematic attempt to find summation theorems for basic Appell series. Sharma and Jain [6] showed that q-Appell functions can be brought within the purview of Lie-theory by deriving reduction formulae for q-Appell functions namely Φ^1 and Φ^2 using the dynamical symmetry algebra of basic hypergeometric function $_2\Phi_1$.

Definitions and preliminaries:

In this chapter, we use the following definitions and preliminaries based on the text by Gasper and Rehman [7] which are given as follows:

$(a; q)_n = (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1})$

And $(a)_\infty = \lim_{n\to\infty}(a)_n$

$\lim_{q\to 1}(q;q)_n = (1)_n = n!$, and $(a; q)_0 = 1$

$$\phi^1[a; b; b'; c; x; y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(q)_m(q)_n(c)_{m+n}}x^m y^n$$

$$\Phi^2[a;b;b';c;c',x;y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(q)_m(q)_n(c)_m(c')_n}x^m y^n$$

$$\Phi^3[a;a';b;b';c;x;y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(a')_n(b)_m(b')_n}{(q)_m(q)_n(c)_{m+n}}x^m y^n$$

$$\Phi^4[a;b;c;c';x,y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(q)_m(q)_n(c)_m(c')_n}x^m y^n$$

withsame convergence regions as for the ordinary Appell's hypergeometric functions, and with even sharper convergence when $|q|< 1$, [1].

2. INTEGRAL REPRESENTATION FOR BASIC APPELL'S HYPERGEOMETRIC FUNCTIONS

In this section, by using the definition of q-integral, we express the basic Appell functions Φ^1,Φ^2, Φ^3 in terms of double integrals which takes the form of ordinary Appell's hypergeometric functions as a limiting case.

Theorem (2.1): $\phi^{2}[a;b;b';c;c',x;y]=\frac{\Gamma_q(c)\Gamma_q(c')}{\Gamma_q(b)\Gamma_q(b')\Gamma_q(c-b)\Gamma_q(c'-b')}\times$

$$\int_0^1\int_0^1 u^{b-1}v^{b'-1}(1-uq)^{c-b-1}(1-vq)^{c'-b'-1}(1-ux-vy)^{-a}\,d_q(u)d_q(v),$$

taken over the triangle, $u \geq 0,\ v \geq 0,\ u+v \leq 1$.

Proof: Taking L.H.S of theorem (2.1), we have

$$\Phi^2[a;b;b';c;c'x;y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(q)_m(q)_n(c)_m(c')_n}x^m y^n \quad (2.1)$$

Since, $(a;q)_m = \frac{\Gamma_q(a+m)}{\Gamma_q(a)}(1-q)^m$

Therefore,

$$\frac{(b')_n(b)_m}{(c)_m(c')_n} = \frac{\Gamma_q(c)\Gamma_q(c')\Gamma_q(b+m)\Gamma_q(b'+n)}{\Gamma_q(b)\Gamma_q(b')\Gamma_q(c+m)\Gamma_q(c'+n)}$$

$$=\frac{\Gamma_q(c)\Gamma_q(c')\Gamma_q(b+m)\Gamma_q(b'+n)}{\Gamma_q(b)\Gamma_q(b')\Gamma_q(c+m)\Gamma_q(c'+n)}\cdot\frac{\Gamma_q(c-b)\Gamma_q(c'-b')}{\Gamma_q(c-b)\Gamma_q(c'-b')}$$

$$=\frac{\Gamma_q(c)\Gamma_q(c')}{\Gamma_q(c-b)\Gamma_q(c'-b')\Gamma_q(b)\Gamma_q(b')}\quad B(b+m,c-b)\quad B(b'+n,c'-b')$$

Now making use of q-beta function, we have

$$\frac{(b')_n(b)_m}{(c)_m(c')_n}=\frac{\Gamma_q(c)\Gamma_q(c')}{\Gamma_q(c-b)\Gamma_q(c'-b')\Gamma_q(b)\Gamma_q(b')}\times$$

$$\int_0^1 u^{b+m-1}(1-uq)^{c-b-1}\,d_q(u)\int_0^1 v^{b'+n-1}(1-vq)^{c'-b'-1}\,d_q(v)$$

$$=\frac{\Gamma_q(c)\Gamma_q(c')}{\Gamma_q(c-b)\Gamma_q(c'-b')\Gamma_q(b)\Gamma_q(b')}\times$$

$$\int_0^1\int_0^1 u^{b+m-1}(1-uq)^{c-b-1}(1-vq)^{c'-b'-1}\,v^{b'+n-1}d_q(u)d_q(v).$$

Using above result in (2.1), we get

$$^{2}[a;b;b';c;c'x;y]=\frac{\Gamma_q(c)\Gamma_q(c')}{\Gamma_q(c-b)\Gamma_q(c'-b')\Gamma_q(b)\Gamma_q(b')}\times$$

$$\int_0^1\int_0^1 u^{b-1}v^{b'-1}(1-uq)^{c-b-1}(1-vq)^{c'-b'-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}}{(q)_m(q)_n}(ux)^m(vy)^n d_q(u)d_q(v).$$

Using series manipulation in above equation, we get

$$\phi^{2}[a;b;b';c;c'x;y]=\frac{\Gamma_q(c)\Gamma_q(c')}{\Gamma_q(c-b)\Gamma_q(c'-b')\Gamma_q(b)\Gamma_q(b')}\int_0^1\int_0^1 u^{b-1}v^{b'-1}(1-uq)^{c-b-1}(1-$$

vqc′−b′−1m=0∞n=0m(a)m(q)m(q)n(ux)m−n(vy)ndqudq(v).

Thus, we get

$$\phi^{2}[a;b;b';c;c'x;y]=\frac{\Gamma_q(c)\Gamma_q(c')}{\Gamma_q(c-b)\Gamma_q(c'-b')\Gamma_q(b)\Gamma_q(b')}\times$$

$$\int_0^1\int_0^1 u^{b-1}v^{b'-1}(1-uq)^{c-b-1}(1-vq)^{c'-b'-1}(1-ux-vy)^{-a}\,d_q(u)d_q(v).$$

This completes the proof of the theorem.

Similarly we can prove the following theorems

Theorem (2.2):

$$\frac{\Gamma_q(b)\Gamma_q(b')\Gamma_q(c-b-b')}{\Gamma_q(c)}\phi^{1}[a;b;b';c;x;y]$$

$$=\int_0^1\int_0^1 u^{b-1}v^{b'-1}(1-u-v)^{c-b-b'-1}(1-ux-vy)^{-a}\,d_q(u)d_q(v)$$

Taken over the triangle $u\geq 0,\ v\geq 0,\ u+v\leq 1.$

Theorem (2.3):

$$\frac{\Gamma_q(b)\Gamma_q(b')\Gamma_q(c-b-b')}{\Gamma_q(c)}\phi^{3}[a;a';b;b';c;x;y]$$

$$=\int_0^1\int_0^1 u^{b-1}v^{b'-1}(1-u-v)^{c-b-b'-1}(1-ux)^{-a}(1-vy)^{-a'}d_q(u)d_q(v).$$

Theorem (2.4):

$$\frac{\Gamma_q(a)\Gamma_q(c-a)}{\Gamma_q(c)}\ \phi^{1}[a;b;b';c;x;y]$$

$$=\int_0^1 u^{a-1}(1-uq)^{c-b-1}(1-ux)^{-b}(1-ux)^{-b'}d_q(u).$$

3. TRANSFORMATION FORMULAE FOR THE Q-APPELL FUNCTIONS:

In this section we have derived certain transformation formulae for the q-Appell functions.

Theorem (3.1):

$$\phi^{1}[a;b;b';c;x;y]=(1-y)^{-b'}\phi^{3}\left[a;\ c-a;\ b,b';c,x,\frac{-y}{1-y}\right]$$

Proof: Taking the L.H.S of theorem (3.1), we have

$$\phi^1[a;b;b';c;x;y]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(q)_m(q)_n(c)_{m+n}}x^m y^n$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(a+m)_n(b)_m(b')_n}{(q)_m(q)_n(c)_m(c+m)_n}x^m y^n$$

$$=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(q)_m(c)_m}x^m\sum_{n=0}^{\infty}\frac{(a+m)_n(b')_n}{(q)_n(c+m)_n}y^m$$

$$=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(q)_m(c)_m}x^m{}_2\phi_1[b',a+m,c+m,y]$$

$$=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(q)_m(c)_m}x^m(1-y)^{-b'}\ 2\phi 1\left[b',c-a,c+m,\frac{-y}{1-y}\right].$$

Hence,

$$\phi^1[a;b;b';c;x;y]=(1-y)^{-b'}\phi^3\left[a;\ c-a;\ b,b';c,x,\frac{-y}{1-y}\right]$$

This completes the proof.

Similarly we can prove the following theorems

Theorem (3.2):

$$\phi^2[a;b;b';c,c';x;y]=(1-x)^{-a}\phi^2\left[a;c-b,b';c,c';\frac{-x}{1-x},y\right]$$

Theorem (3.3):

$$(1-y)^{-b'}\phi^2\left[a;b,b';c,a;x,\frac{-y}{1-y}\right]$$

$$=\phi^1[b;a-b';b';c;x;x(1-y)]$$

Theorem (3.4):

$$\phi^1[a;b;b';c;x;y]=(1-x)^{-b}(1-y)^{-b'}\times$$

$$\phi^1\left[c-a;b,b';c;\frac{-x}{1-x},\frac{-y}{1-y}\right].$$

4. SPECIAL CASES

In this section, we discuss some of the special cases of the main results established in the previous section. As q→1, the above results take the form of well-known transformation formulae of classical Appell functions F^1, F^2, F^3 in terms of definite integrals. The formulae given by [8] are as follows

4.1. $F^2[a;b;b';c;c';x;y]=\frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(b')\Gamma(c-b)\Gamma(c'-b')}\times\int_0^1\int_0^1 u^{b-1}v^{b'-1}(1-u)^{c-b-1}(1-v)^{c'-b'-1}(1-ux-vy)^{-a}\ d(u)d(v).$

4.2. $F^1[a;b;b';c;x;y]=\frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')}\times\int_0^1\int_0^1 u^{b-1}v^{b'-1}(1-u-v)^{c-b-b'-1}(1-ux-vy)^{-a}\,d(u)d(v).$

4.3. $F^3[a;b;b';c;x;y]=\frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')}\times\int_0^1\int_0^1 u^{b-1}v^{b'-1}(1-u-v)^{c-b-b'-1}(1-ux)^{-a}(1-vy)^{-a'}d(u)d(v).$

5. CONCLUSION

In this paper we have explored the possibility for derivation of some integral representation and transformation formulae for basic hypergeometric functions of one and more variables, in particular the q-Appell functions, using certain fundamental tools of q-fractional calculus. The results thus derived are general in character and likely to find certain applications in the theory of basic hypergeometric functions. In this connection one can refer to the work of Baily [8] and Sharma [6]

REFERENCES

[1]. Jackson, F. H (1910), On basic double Hyper geometric function, Quart, J. Pure and Appl. Math; 41, 193-203.

[2]. Agrawal, R. P (1969), Certain fraction q-integrals and q-derivatives. Proc. Edin. Math. Soc., 66, 365-370.

[3]. George E. Andrews. (1972), Basic Hypergeometric series, J. London Math. Soc. (2) 4 618.

[4]. Yadav R.K and Purohit S.D (2006), on fractions q-derivatives and transformation of the generalized basic hypergeometric functions, Indian Acad. Math.2, 321-326.

[5]. Slater, L.J (1966), Generalized Hypergeometric function, Cambridge University, Press.

[6]. Sharma. K and Jain. R (2007), Lie-theory and q-Appell functions, Proceedings of the National Academy of Science India, 77, 259-262.

[7]. Gasper .G and Rahman.M (1990), Basic hypergeometric series, Cambridge University Press.

[8]. Baily, W. N (1961), Generalized hypergeometric series, Cambridge University, Press.